Lec 1

Math 2

Sequences & Series

Sequences :

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

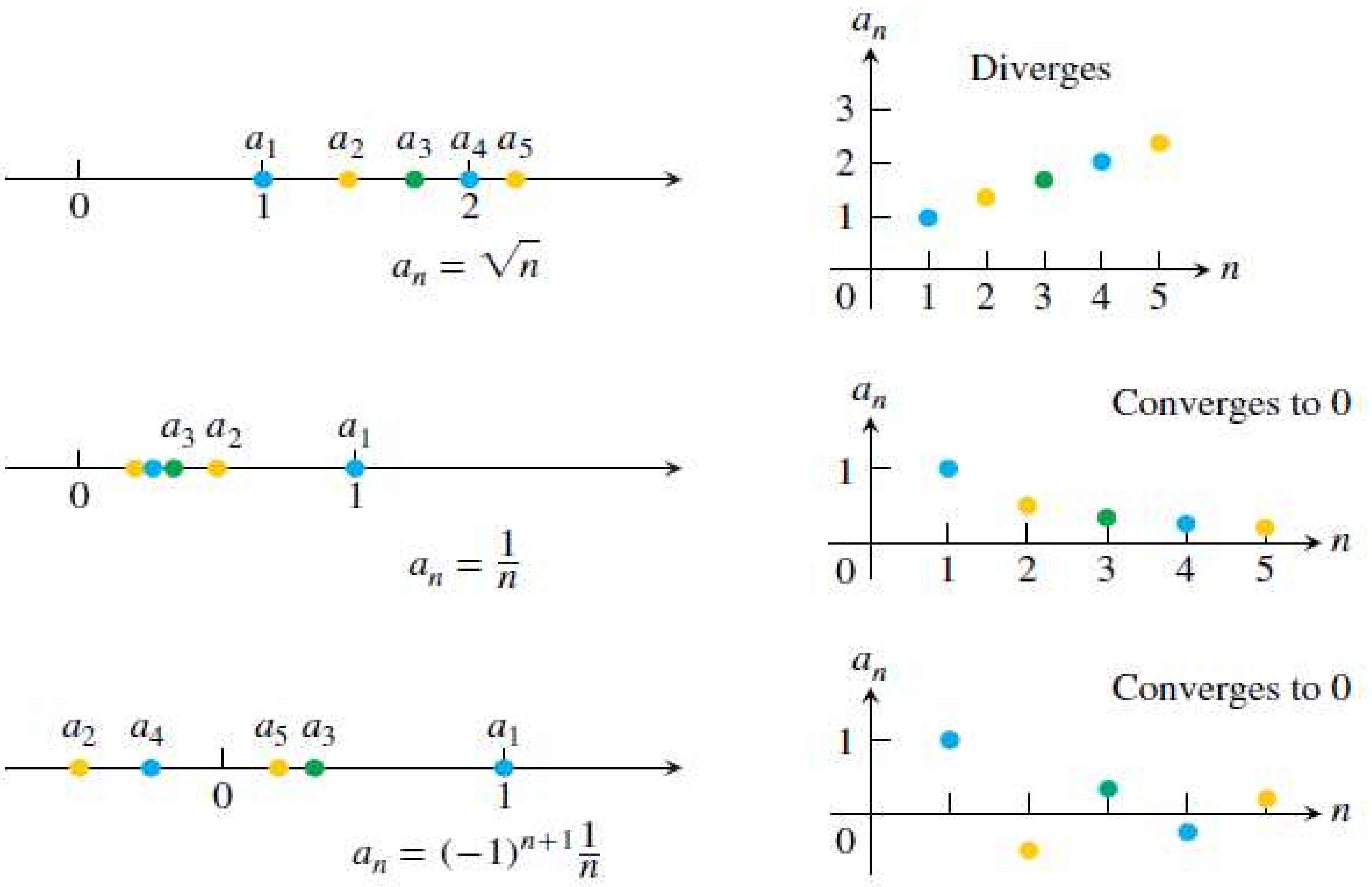
EX:

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$

$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}$$

$$\{d_n\} = \{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}.$$



a_1 a_2 a_3 a_4 a_5
0 1 2
$a_n = \sqrt{n}$
Diverges
a_n
3
2
1
0 1 2 3 4 5 n
a_3 a_2 a_1
0 1
$a_n = \frac{1}{n}$
Converges to 0
a_n
1
0 1 2 3 4 5 n
a_2 a_4 a_5 a_3 a_1
0 1
$a_n = (-1)^{n+1}\frac{1}{n}$
Converges to 0
a_n
1
0
n

DEFINITIONS Converges, Diverges, Limit

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 11.2).

EXAMPLE 1 Applying the Definition

Show that

(a) $\lim_{n\to\infty} \frac{1}{n} = 0$ **(b)** $\lim_{n\to\infty} k = k$ (any constant k)

Solution

(a) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \epsilon.$$

This implication will hold if $(1/n) < \epsilon$ or $n > 1/\epsilon$. If N is any integer greater than $1/\epsilon$, the implication will hold for all $n > N$. This proves that $\lim_{n\to\infty}(1/n) = 0$.

(b) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive integer for N and the implication will hold. This proves that $\lim_{n\to\infty} k = k$ for any constant k. ■

EXAMPLE 2 A Divergent Sequence

Show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ diverges.

Solution Suppose the sequence converges to some number L. By choosing $\epsilon = 1/2$ in the definition of the limit, all terms a_n of the sequence with index n larger than some N must lie within $\epsilon = 1/2$ of L. Since the number 1 appears repeatedly as every other term of the sequence, we must have that the number 1 lies within the distance $\epsilon = 1/2$ of L. It follows that $|L - 1| < 1/2$, or equivalently, $1/2 < L < 3/2$. Likewise, the number -1 appears repeatedly in the sequence with arbitrarily high index. So we must also have that $|L - (-1)| < 1/2$, or equivalently, $-3/2 < L < -1/2$. But the number L cannot lie in both of the intervals $(1/2, 3/2)$ and $(-3/2, -1/2)$ because they have no overlap. Therefore, no such limit L exists and so the sequence diverges.

Theorem:

$$\lim_{n\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{f'(x)}{g'(x)} \quad \text{(L'Hopital's Rule)}$$

يستخدم في حالة التعويض وينتج $\frac{0}{\infty}$., $\frac{\infty}{\infty}$, $\frac{0}{0}$

EX: Use L'Hopital's Rule to find

$$\lim_{n\to\infty} \frac{2^n}{5n}.$$

By l'Hôpital's Rule (differentiating with respect to n),

$$\lim_{n\to\infty} \frac{2^n}{5n} = \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{5}$$
$$= \infty.$$

Applying L'Hôpital's Rule to Determine Convergence

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

Solution The limit leads to the indeterminate form 1^{∞}. We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n$$

$$= n \ln\left(\frac{n+1}{n-1}\right).$$

Then,

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \qquad \infty \cdot 0$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \qquad \frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \qquad \text{l'Hôpital's Rule}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to e^2.

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Ex: Check the convergence of the following sequences :

1- $a_n = \sqrt{\frac{n+1}{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{n}} = \sqrt{\lim_{n\to\infty} (1+\frac{1}{n})} = \sqrt{1+\frac{1}{\infty}} = \sqrt{1+0}$$

$$= 1\ (conv.)$$

2-$a_n = (1-\frac{3}{x})^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1-\frac{3}{x})^n = \lim_{n\to\infty} (1+\frac{-3}{x})^n = e^{-3}$$ conv. (from 5)

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also
$\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots,$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots.$$

If $r = 1$, the nth partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of a. If $r = -1$, the series diverges because the nth partial sums alternate between a and 0. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$ Multiply s_n by r.

$$s_n - rs_n = a - ar^n$$ Subtract rs_n from s_n. Most of the terms on the right cancel.

$$s_n(1 - r) = a(1 - r^n)$$ Factor.

$$s_n = \frac{a(1 - r^n)}{1 - r}, \qquad (r \neq 1).$$ We can solve for s_n if $r \neq 1$.

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ (as in Section 11.1) and $s_n \to a/(1 - r)$. If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Ex: $\sum_{n=0}^{\infty}(\frac{1}{2})^n$ is a G.S.

a=1 ,$r=\frac{1}{2} < 1$ $\therefore$ converge to $\frac{1}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$

EX: $\sum_{n=0}^{\infty} 3^n$ divergence series because r=3>1

Express the repeating decimal 5.232323 ... as the ratio of two integers.

Solution We look for a pattern in the sequence of partial sums that might lead to a formula for s_k. The key observation is the partial fraction decomposition

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and

$$s_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

Removing parentheses and canceling adjacent terms of opposite sign collapses the sum to

$$s_k = 1 - \frac{1}{k+1}.$$

We now see that $s_k \to 1$ as $k \to \infty$. The series converges, and its sum is 1:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Tests of convergences :

nth term test for divergence :

for series $\sum_{n=1}^{\infty} a_n$ if $\lim\limits_{n\to\infty} a_n \neq 0$ then the series is divergence

but $\lim\limits_{n\to\infty} a_n = 0$ then this doesn't mean that $\sum a_n$ is converge .

EX:

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0.$

The integral test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \le 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1},$$

$b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$.

the series converges by the Integral Test. We emphasize that the sum of the p-series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p} \lim_{b\to\infty} (b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

We have convergence for $p > 1$ but divergence for every other value of p.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by the Integral Test. The function $f(x) = 1/(x^2 + 1)$ is positive, continuous, and decreasing for $x \geq 1$, and

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2 + 1}\,dx &= \lim_{b\to\infty} \Big[\arctan x\Big]_1^b \\ &= \lim_{b\to\infty} [\arctan b - \arctan 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.\end{aligned}$$

EX:

$\sum_{n=2}^{\infty} \frac{1}{nlnn}$,,,, f(x)= $\frac{1}{xlnx}$

$\int_2^{\infty} f(x)dx = \lim_{n\to\infty} (\int_2^{\infty} \frac{1}{xlnx} dx) = \lim_{n\to\infty} (\ln(lnx)]_2^n =$

$\lim_{n\to\infty} (lnlnn - lnln2) = \infty$

$\therefore \sum_{n=2}^{\infty} \frac{1}{nlnn}$ is diverges

The ratio test :

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho .$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

(a) For the series $\sum_{n=0}^{\infty}(2^n + 5)/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right) \rightarrow \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \rightarrow 4.$$

The root test :

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho .$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

$\sum_{n=1}^{\infty}(1-\frac{3}{n})^{7n^2}$

$$\lim_{n\to\infty} \sqrt[n]{(1-\frac{3}{n})^{7n^2}} = \lim_{n\to\infty}(1-\frac{3}{n})^{7n} =$$

$(e^{-3})^7 = e^{-21} < 1$

Alternating Series :

A series of form $\sum_{n=0}^{\infty}(-1)^n a_n$ is called **Alternating Series** i.e.

$\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 - \cdots \dots.$

or $\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty}(\cos n\pi)\, a_n$

The Alternating Series Test :

The series $\sum_{n=0}^{\infty}(-1)^n a_n$ is convergence if :

1. $a_n > 0$ $(a_n \ is\ positive\ )$
2. $a_n \geq a_{n+1}$ for all n $\geq N$,for some integer N
3. $\lim_{n\to\infty} a_n = 0$

Ex:

1 $-\sum_{n=0}^{\infty}(-1)^n \frac{1}{n}$ is converge since $\lim_{n\to\infty} \frac{1}{n} = 0$

2. $\sum_{n=0}^{\infty} \frac{(\cos n\pi)}{1+n^2} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{1+n^2}$ is converge since $\lim_{n\to\infty} \frac{1}{1+n^2} = 0$

Note:

1. If $\sum|(-1)^n a_n|$ is converge then $\sum(-1)^n a_n$ is converge

If $\sum(-1)^n a_n$ is diverge then $\sum|(-1)^n a_n|$ is also diverge

The Absolutely & Conditional Convergence:

1. If $\sum(-1)^n a_n$ is convergence .this series is called **Absolutely Convergent** if $\sum|(-1)^n a_n|$ is converge.
2. If $\sum(-1)^n a_n$ is convergence and $\sum|(-1)^n a_n|$ is divergence then $\sum(-1)^n a_n$ is called **Conditional Convergent**

1. $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$ is conv. but $\sum\left|\frac{(-1)^n}{n}\right| = \sum_{n=0}^{\infty}\frac{1}{n}$ is diverge

$\sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$ is **Conditionally Convergent**

$2 - \sum_{n=0}^{\infty}(-1)^n\frac{n}{n+1}$ is divergence because $\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$

Power Series :

This has the form $\sum_{n=1}^{\infty} a_n(x-h)^n \;\; = a_1(x-h) + a_2(x-h)^2 + \;\; a_3(x-h)^3 \ldots\ldots..$

To study these series we find the interval of x for absolute convergence by using the ratio test .

EX: Find the interval of absolute convergence of :

1. $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$

Using ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| < 1 \quad = \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| < 1$$

=0<1 for every value of x

$\therefore$ interval of conv. is $\quad \infty > x > -\infty$

$\sum_{n=0}^{\infty} 3^n \frac{(x+5)^n}{4^n}$

Using ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\lim_{n\to\infty} \left| 3^{n+1} \frac{(x+5)^{n+1}}{4^{n+1}} . \ . \frac{4^n}{3^n (x+5)^n} \right| < 1 \quad = \lim_{n\to\infty} \left|\frac{3}{4}(x+5)\right| < 1$

= $-1 < \frac{3}{4}(x+5) < 1$

$\frac{-4}{3} <$ x+5 $< \frac{4}{3}$

$-\frac{19}{3} < x < \frac{-11}{3}$ radius of conv. R= 4/3

DEFINITIONS Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

EXAMPLE 1 Finding a Taylor Series

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$
\begin{aligned}
f(x) &= x^{-1}, & f(2) &= 2^{-1} = \frac{1}{2}, \\
f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{2^2}, \\
f''(x) &= 2!x^{-3}, & \frac{f''(2)}{2!} &= 2^{-3} = \frac{1}{2^3}, \\
f'''(x) &= -3!x^{-4}, & \frac{f'''(2)}{3!} &= -\frac{1}{2^4}, \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^n n! x^{-(n+1)}, & \frac{f^{(n)}(2)}{n!} &= \frac{(-1)^n}{2^{n+1}}.
\end{aligned}
$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

Differential Equations

A *differential equation* is a relationship between an independent variable, x, a dependent variable y, and one or more derivatives of y with respect to x.

e.g. $$x^2\frac{dy}{dx} = y\sin x = 0$$

$$xy\frac{d^2y}{dx^2} + y\frac{dy}{dx} + e^{3x} = 0$$

Types : 1- Ordinary (O.D.E) If (D.E) involves only a single independent variable this derivatives are called ordinary derivatives, and the equation is called ordinary (D.E)

Ex:

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx}$$

2-Partial (P.D.E) If there are two or more independent variables derivatives are called partial derivatives, and the equation is called partial(D.E)

Ex:

$$\frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial r^2} + \cdots \ldots .$$

Order : Is that the derivative of highest order in the equation for example:

$x\frac{dy}{dx} - y^2 = 0$ is an equation of the 1st order

$xy\frac{d^2y}{dx^2} - y^2 \sin x = 0$ is an equation of the 2nd order

$\frac{d^3y}{dx^3} - y\frac{dy}{dx} + e^{4x} = 0$ is an equation of the 3rd order

Degree: Is the higher degree of higher derivatives .

1. $y''^3 + y'^5 = \sin x$ (order 2 degree 3)
2. $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = e^x$ (order 2 degree 1)

The Solution of First Order D.E.:

Separable :قابلة للفصل

Let us consider equations of the form $\frac{dy}{dx} = f(x).F(y)$ and of the form $\frac{dy}{dx} = \frac{f(x)}{F(y)}$, i.e. equations in which the right-hand side can be expressed as products or quotients of functions of x or of y.

A few examples will show how we proceed.

Solve $\frac{dy}{dx} = \frac{2x}{y+1}$

We can rewrite this as $(y+1)\frac{dy}{dx} = 2x$

Now integrate both sides with respect to x:

$$\int (y+1)\frac{dy}{dx}\,dx = \int 2x\,dx \quad \text{i.e.}$$

$$\int (y+1)\,dy = \int 2x\,dx$$

and this gives $\frac{y^2}{2} + y = x^2 + C$

Ex:

Solve $\frac{dy}{dx} = \frac{x(1+y^2)}{2-3x^2}$

$$\frac{dy}{(1+y^2)} = \frac{x}{2-3x^2}$$

$$\int \frac{dy}{(1+y^2)} = \int \frac{x}{2-3x^2}\,dx$$

$$\tan^{-1} y = -\frac{1}{6} \ln\left|2-3x^2\right| + c$$

Homogeneous : متجانسة

The differential equation as form

M(x,y) dx+N(x,y) dy =0,

Where M and N are functions of x and y is called (H.d.e) if satisfy the condition

$$\left.\begin{array}{l} M(kx, ky) = k^n M(x, y) \\ N(kx, ky) = k^n N(x, y) \end{array}\right\} \text{Where k is constant.}$$

For example

1- $(x^2 - y^2)dx + 2xydy = 0$

$M = x^2 - y^2$, $N = 2xy$

$M(kx,ky) = (kx)^2 - (ky)^2 = k^2x^2 - k^2y^2 = k^2(x^2 - y^2)$

$k^2(M)$

$N(kx,ky) = 2(k^2xy) = k^2(2xy)$

$k^2(N)$.

The equation is (H.d.e).

3- Solve $(x-y)dx + xydy = 0$

$M = x - y$, $N = xy$

$M(kx,ky) = (kx) - (ky) = k(x - y)$

$k(M)$

$N(kx,ky) = k^2(xy)$

$k^2(N)$.

The equation is not (H.d.e).

method:-

1. Put $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ 2 $-$Let v= $\frac{y}{x}$

3-Put (1)in (2) == $\frac{dy}{dx} = f(v)$

4-From (2) y=xv, and dy = x dv+vdx, divided by dx

Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

Here, all terms of the RHS are of degree 2, i.e. the equation is homogeneous.

$\therefore$ We substitute $y = vx$ (where v is a function of x)

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

and $\frac{x^2 + y^2}{xy} = \frac{x^2 + v^2x^2}{vx^2} = \frac{1 + v^2}{v}$

The equation now becomes:

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1 + v^2}{v} - v$$

$$= \frac{1 + v^2 - v^2}{v} = \frac{1}{v}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1}{v}$$

Now you can separate the variables and get the result in terms of v and x.

$$\frac{v^2}{2} = \ln x + c$$

Because

$$\int v \, dv = \int \frac{1}{x} dx$$

$$\therefore \quad \frac{v^2}{2} = \ln x + C$$

All that remains now is to express v back in terms of x and y. The substitution we used was $y = vx \quad \therefore \quad v = \frac{y}{x}$

$$\therefore \quad \frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln x + C$$

$$y^2 = 2x^2(\ln x + C)$$

Ex: **Solve** $x^2 dy + (y^2 - xy)dx = 0$

$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$ ==== $\frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2}$ **, it is homogeneous , let v=** $\frac{y}{x}$

$\therefore \frac{dy}{dx} = v - v^2$ ==== $x\frac{dv}{dx} + v = v - v^2$

$-\frac{dv}{v^2} = \frac{dx}{x}$ ==== $\int -\frac{dv}{v^2} = \int \frac{dx}{x}$

$$\frac{1}{v} = \ln x + c \quad = \frac{x}{y} = \ln x + c$$

Exact : التامة

_The differential equation as form

M(x,y) dx+N(x,y) dy =0 …(*)   with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is exact . To solve (*) we integrate

$\int M(x,y)\,dx + \int N(x,y)\,dy = c$

Ex: Solve$(y^2 e^{xy^2} + 4x^3)\,dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$

$$\frac{\partial M}{\partial y} = y^2 e^{xy^2}(2xy) + e^{xy^2}(2y) = e^{xy^2}(2xy^3 + 2y)$$

$$\frac{\partial N}{\partial x} = 2xye^{xy^2}\left(y^2\right) + e^{xy^2}(2y) = e^{xy^2}(2xy^3 + 2y)$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (Eq. is exact)

M(x ,y) dx+ N(x ,y) d y =0 ===== $\int M(x,y)\,dx + \int N(x,y)\,dy$=c

= $\int (y^2 e^{xy^2} + 4x^3)\,dx + \int \left(2xye^{xy^2} - 3y^2\right) dy$=c

$$e^{xy^2} + 4\frac{x^4}{4} + e^{xy^2} - 3\frac{y^3}{3} = c \text{======} \quad 2e^{xy^2} + x^4 - y^3 = c$$

Linear: الخطية

Is D.E. of the form $\frac{dy}{dx} + P(x)y = Q(x)$,where P and Q are functions of x . To solve it ,multiply both sides by integrating factor

$$I.Fe^{\int p(x)dx}$$

Solve $x\frac{dy}{dx} + y = x^3$

First we divide through by x to reduce the first term to a single $\frac{dy}{dx}$

i.e. $\frac{dy}{dx} + \frac{1}{x}.y = x^2$

Compare with $\left[\frac{dy}{dx} + Py = Q\right]$ $\therefore$ $P = \frac{1}{x}$ and $Q = x^2$

$IF = e^{\int P dx}$ $\int P\,dx = \int \frac{1}{x}\,dx = \ln x$

$\therefore$ $IF = e^{\ln x} = x$ $\therefore$ $IF = x$

The solution is $y.IF = \int Q.IF\,dx$

so $yx = \int x^2.x\,dx = \int x^3\,dx = \frac{x^4}{4} + C$ $\therefore$ $xy = \frac{x^4}{4} + C$

Solve $\frac{dy}{dx} + y \cot x = \cos x$

Compare with $\left[\frac{dy}{dx} + Py = Q\right] \quad \therefore \begin{cases} P = \cot x \\ Q = \cos x \end{cases}$

$IF = e^{\int P dx} \qquad \int P\,dx = \int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln \sin x$

$\therefore \; IF = e^{\ln \sin x} = \sin x$

$y.IF = \int Q.IF\,dx \quad \therefore \; y \sin x = \int \sin x \cos x\,dx = \frac{\sin^2 x}{2} + C$

$\therefore \; y = \frac{\sin x}{2} + C \operatorname{cosec} x$

Bernoulli's Equation :

These are equations of the form:

$$\frac{dy}{dx} + Py = Qy^n$$

where, as before, P and Q are functions of x (or constants).

The trick is the same every time:

(a) Divide both sides by y^n. This gives:

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

(b) Now put $z = y^{1-n}$

so that, differentiating, $\frac{dz}{dx} = \ldots\ldots\ldots \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

So we have:

$$\frac{dy}{dx} + Py = Qy^n \qquad (1)$$

$$\therefore\ y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \qquad (2)$$

Put $z = y^{1-n}$ so that $\dfrac{dz}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$

If we now multiply (2) by $(1-n)$ we shall convert the first term into $\dfrac{dz}{dx}$.

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)Py^{1-n} = (1-n)Q$$

Remembering that $z = y^{1-n}$ and that $\dfrac{dz}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$, this last line can now be written $\dfrac{dz}{dx} + P_1 z = Q_1$ with P_1 and Q_1 functions of x.

This we can solve by use of an integrating factor in the normal way.

Finally, having found z, we convert back to y using $z = y^{1-n}$.

Solve $\frac{dy}{dx} + \frac{1}{x}y = xy^2$

(a) Divide through by y^2,

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}.y^{-1} = x$$

(b) Now put $z = y^{1-n}$, i.e. in this case $z = y^{1-2} = y^{-1}$

$$z = y^{-1} \quad \therefore \quad \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$$

(c) Multiply through the equation by (-1), to make the first term $\frac{dz}{dx}$.

$$-y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -x$$

so that $\frac{dz}{dx} - \frac{1}{x}z = -x$ which is of the form $\frac{dz}{dx} + Pz = Q$ so that you can now solve the equation by the normal integrating factor method. What do you get?

$$y = (Cx - x^2)^{-1}$$

Solve $2y - 3\frac{dy}{dx} = y^4 e^{3x}$

$$2y - 3\frac{dy}{dx} = y^4 e^{3x}$$

$$\therefore \frac{dy}{dx} - \frac{2}{3}y = -\frac{y^4 e^{3x}}{3}$$

$$\therefore y^{-4}\frac{dy}{dx} - \frac{2}{3}y^{-3} = -\frac{e^{3x}}{3}$$

Put $z = y^{1-4} = y^{-3}$ $\quad \therefore \frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$

Multiplying through by (-3), the equation becomes:

$$-3y^{-4}\frac{dy}{dx} + 2y^{-3} = e^{3x}$$

i.e. $\frac{dz}{dx} + 2z = e^{3x}$

$\text{IF} = e^{\int P\,dx}$ $\quad \int P\,dx = \int 2\,dx = 2x \quad \therefore \text{IF} = e^{2x}$

$$\therefore ze^{2x} = \int e^{3x}e^{2x}\,dx = \int e^{5x}\,dx$$

$$= \frac{e^{5x}}{5} + C$$

But $z = y^{-3}$ $\quad \therefore \frac{e^{2x}}{y^3} = \frac{e^{5x} + A}{5}$

$$\therefore y^3 = \frac{5e^{2x}}{e^{5x} + A}$$

The Second Order (D.E.): F(x, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$)

1. **Special Type**: حالة خاصة (reduced to first order)

The equation of the form F(x, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$) can be reduced to first order by suppose that :

$$P = \frac{dy}{dx} \qquad \frac{dp}{dx} = \frac{d^2y}{dx^2}$$

Then the equation F(x, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$) takes form F(x, P, $\frac{dp}{dx}$) which is first order (D.E.)

EX:

Solve the following differential equation

$$X^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = 1$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 1/x^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (**)$$

Let $p = \frac{dy}{dx}$ and $\frac{d^2y}{dx^2} = \frac{dp}{dx}$, put in (**)

$$\frac{dp}{dx} + \frac{1}{x}p = 1/x^2$$, which linear in p,

$I = e^{\int p dx} = I = e^{\int dx/x} = x.$

$Ip = \int IQ\ dx + c \Rightarrow Ip = \int x(1/x^2)\,dx + c = \ln x + c$

$\therefore\ xp = \ln x + c$

$P = (\ln x)/x + c/x$

Let $\frac{dy}{dx} = (\ln x)/x + c/x$

$dy = [(\ln x)/x]dx + (c/x)dx,$

$y = (\ln x)^2/2 + c \ln x + c_1$

Homogeneous : متجانسة

The equation of the form :

$ay'' + by' + cy = 0$ or (*)

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ or

$aD^2y + bDy + cy = 0$ where a,b,c are constant and $D = \frac{d}{dx}$ is called

How to solve this equation we shall now find how to determine m such that

$y=e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$ put in(*)

$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$

$e^{mx}(am^2 + bm + c) = 0$

Since $e^{mx} \neq 0$ $(am^2 + bm + c) = 0$ (**) which is called the characteristic equation

Then we say that y=e^{mx} is the solution of (*) ↔ m is the root of (**)

The general solution of (*) there is three cases of root :

If $m_1 \neq m_2$ in equation (**) , the solution of the homogenous equation (*) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

If $m_1 = m_2$ in equation (**) , the solution of the homogenous equation (*) is

$$c_1 e^{mx} + c_2 x e^{mx}$$

If m_1 and m_2 roots ($m_1 = \alpha + \beta i$,,, $m_2 = \alpha - \beta i$) in equation (**) , the solution of the homogenous equation (*) is

$$y = e^{\alpha x}(c_1 sin\beta x + c_2 cos\beta x)$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

Auxiliary equation: $m^2 + 5m + 6 = 0$

$\therefore\ (m+2)(m+3) = 0 \quad \therefore\ m = -2$ or $m = -3$

$\therefore$ Solution is $y = Ae^{-2x} + Be^{-3x}$

Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

Auxiliary equation: $m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0 \quad \therefore\ m = -2$ (twice)

The solution is: $y = e^{-2x}(A + Bx)$

Non homogenous (D.E): غيرمتجانسة

Consider the equation of the form :

$ay'' + by' + cy = f(x) \qquad (\#)$

where a,b,c are constant is a non-**homogeneous (D.E.)** of second order

To find the general solution of (#)

1. We find the solution of homogenous part y_h of eq. (#)
2. We find any another special solution y_p eq.(#)

Then the general solution of (#) is $y = y_h + y_p$

Un determined Coefficients Method:

We have seen that to find the particular integral, we assume the general form of the function on the RHS of the equation and determine the values of the constants by substitution in the whole equation and equating coefficients. These will be useful:

F(x)	y_p
e^{ax}	Ke^{ax} when e^{ax} not found in y_h Kxe^{ax} when e^{ax} is found in y_h(only once) K$x^2e^{ax}$when e^{ax} found in y_h(twice)
x^n	A$x^n + Bx^{n-1} + \cdots..K$
sin ax cos ax	Asin ax+Bcos ax

EX: Solve $y'' - 2y' + y = 3e^{2x} - 5e^{4x}$ (1)

Sol:

1. We find y_h i.e. $y'' - 2y' + y = 0$

$\therefore m^2 - 2m + 1 = 0 \ \rightarrow \ (m-1)^2 = 0 \ \rightarrow m=1$

$$\therefore y_h = c_1e^x + c_2xe^x$$

$$let\ y_p = Ke^{2x} + He^{4x}$$

$y_p'=2Ke^{2x}+4He^{4x}$

$y_p''=4Ke^{2x}+16He^{4x}$ put in(1) we get $y_p''-2y_p'+y_p=3e^{2x}-5e^{4x}$

$\therefore\ 4Ke^{2x}+16He^{4x}-2(2Ke^{2x}+4He^{4x})+\ Ke^{2x}+He^{4x}=3e^{2x}-5e^{4x}$

$Ke^{2x}+9He^{4x}=3e^{2x}-5e^{4x}$ → K=3 ,H= -5/9

$\therefore\ y_p=3e^{2x}-5/9He^{4x}$

∴y $=y_h+y_p$ →y $=c_1e^x+c_2xe^x+3e^{2x}-5/9He^{4x}$

Ex: solve $y''-4y'-5y=2\sin 2x$ (*)

Sol:

1-We find y_h i.e. $y''-4-5y=0$

$\therefore\ m^2-4m-5=0$ → (m-5)(m+1)= 0 → m_1=5,m_2=-1

$$\therefore y_h = c_1 e^{5x} + c_2 e^{-x}$$

Let y_p= A sin 2x + B cos 2x

y_p' = 2Acos 2x – 2B sin 2x

y_p'' = -4 A sin 2x – 4 B cos 2x put in (*)we get $y_p'' - 4y_p' - 5y_p$= 2 sin2x

$(-4$ A sin 2x – 4 B cos 2x) $-4($2Acos 2x – 2B sin 2x) $-5($A sin 2x + B cos 2x)= 2 sin 2x

$(-4\text{A} + 8\text{B} - 5\text{A})$ sin 2x+(-4 B-8A-5B)= 2 sin2x

Sin 2x: 8B-9A=2

Cos 2x:-9B-8A=0 we solve this eq. to find A and B

$A = \frac{-18}{145}$ $B = \frac{16}{145}$ $\rightarrow$ y_p= $\frac{-18}{145}$sin 2x + $\frac{16}{145}$ cos 2x

$$\therefore\ y = y_h + y_p$$

$y = c_1 e^{5x} + c_2 e^{-x} - \frac{-18}{145}$sin 2x + $\frac{16}{145}$ cos 2x

Variation of Parameters : طريقة تغيير الثوابت

Consider the eq. of the form $ay'' + by' + cy = f(x)$ ….. (#) which is non homogeneous 2nd D.E . to solve (#)

1. We find y_h (solution H-part)

$y_h = c_1 y_1 + c_2 y_2$ where c_1, c_2 are arbitrary constants and y_1 , y_2 are functions which is the solution of (H-part)

1. Let $y_p = v_1 y_1 + v_2 y_2$ (general solution of non H- part) v_1 , v_2 are functions of x i.e. we replace c_1, c_2 by $v_1 and\ v_2$
2. Then these equations are satisfies

$$v_1' y_1 + v_2' y_2 = 0 \quad \text{………(1)}$$

$$v_1' y_1' + v_2' y_2' = \text{f(x)} \quad \text{……..(2)}$$

We find $v_1' and\ v_2'$ by using(Grammer's Rule)

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \qquad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Then y=$y_h + y_p$ (solution of (#))

Ex: Solve $y'' + y = \sec x$

Sol:

$$y = y_h + y_p$$

$y_h = c_1 \sin x + c_2 \cos x$

Let $y_p = v_1 y_1 + v_2 y_2$

$i.e.\ y_p = v_1 sinx + v_2 \cos x$

$\therefore v_1' sinx + v_2' cosx = 0 \ldots\ldots (1)$

$v_1' cosx - v_2' sinx =$ secx …….(2)

$$v_1' = \frac{\begin{vmatrix} 0 & cosx \\ secx & -sinx \end{vmatrix}}{\begin{vmatrix} sinx & cosx \\ cosx & -sinx \end{vmatrix}} = \frac{-cosx\, secx}{-sin^2x - cos^2x} = \frac{-1}{-1} = 1$$

$$v_1 = \int dx \rightarrow v_1 = x$$

$$v_2' = \frac{\begin{vmatrix} sinx & 0 \\ cosx & secx \end{vmatrix}}{\begin{vmatrix} sinx & cosx \\ cosx & -sinx \end{vmatrix}} = \frac{sinxsecx}{-1}$$

$$v_2 = \int \frac{sinxsecx}{-1} dx \rightarrow v_2 = \int \frac{-sinx}{cosx} dx \ = \ln|cosx|$$

$$\therefore y_p = v_1 y_1 \ + v\ _2 y_2 \rightarrow \ y_p = xsinx + \ln|cosx|\cos x$$

Function of many variables:

In many engineering problems we come across function with many variables as an example the volume of a cylinder is a function of radius of the base and the high

V=$\pi r^2 h$.

In general we may defined w=f($x_1, x_2, x_3 \dots . x_n$) $x_1, x_2, x_3 \dots . x_n$ are called the independent variables and w is called dependent variable.

Domain and range:

The domain of w is the set of $x_1, x_2, x_3 \ldots .x_n$ that made w real and defined .

The corresponding w is called the range of function .

Ex:

Function	**domain**	**Range**
W=$\sqrt{y^2 - x^2}$	$y \geq x^2$	$w \geq 0$
W=sin x y	entire plane	-1$\leq w \leq$1
W=ln(x-y)	x> y	-∞ $\leq w \leq$ ∞

Derivative (Partial derivative):

If w=f(x,y) the partial derivative of f with respect that x denoted by $\frac{\partial f}{\partial x}$ or $\frac{\partial w}{\partial x}$ or f_x and is given by

$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$ y is considered have to be a constant when diff. w.r.t. x

Similarly

$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$ x is considered have to be a constant when diff. w.r.t. y

<u>Example:</u>

1. Find f_x and f_y for w=f(x,y)=sin(xy)+y^2x+5

Sol:

f_x =y cos(x y)+ y^2

$$f_y = x\cos(x\ y) + 2xy$$

2. find f_x and f_y for w=f(x,y) = $\frac{2y}{y+cosx}$

sol:

$f_x = \frac{(y+cosx)(0)-2y(-sinx)}{(y+cosx)^2} = \frac{2ysinx}{(y+cosx)^2}$ f_y $H.w$

Second order partial derivatives :

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$ $\frac{\partial^2 f}{\partial y^2} = f_{yy}$ $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

Example: f(x,y)=x cos y+ ye^x

$$f_x = cosy + ye^x$$

$\frac{\partial^2 f}{\partial x^2} = f_{xx} = ye^x$

$$f_y = -siny + e^x$$

$\frac{\partial^2 f}{\partial y^2} = f_{yy} = -cosy$

$\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$ $H.w$

1. if Z=$x^3 + y^4 + x \sin y + e^{xy}$ $find$ Z_{xy} and Z_{yx} at (0,0)

2.If $Z = tan^{-1}\left(\frac{y}{x}\right)$ $then\ show\ that$ $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

Implicit partial derivative:

Assume the equation xy+$z^3x - 2yz$

$= 0$ defineds z as a differential fn. of x and y find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at x=y =z=1

Sol:

$$y+z^3 + 3z^2x\frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial x} = 0$$

$$(3z^2x - 2y)\frac{\partial z}{\partial x} = -y - z^3$$

$$\frac{\partial z}{\partial x} = \frac{-y - z^3}{3z^2x - 2y} = \frac{-1 - 1}{3 - 2} = -2 \qquad \frac{\partial z}{\partial y}\ H.w$$

Chain Rule:

If w=f(x,y) and x=x(t), y=y(t) then w=f(x(t),y(t))=g(t)

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

Also w=f(x,y,z)

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

Example: If f(x,y,z)=2xy+$y^3 + z^2 x$ and x(t)=$\sqrt{t}$,y(t)=$t^2 - 1$,z(t)=$\frac{1}{t}$ find $\frac{df}{dt}$ at t=1.

Sol: at t=1: x=1, y=0,z=1

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

$$f_x = 2\text{y} + z^2 = 0 + 1 = 1$$

$$f_y = 2x + 3y^2 = 2 + 0 = 2$$

$$f_z = 2zx = 2$$

$$\dot{x} = \frac{1}{2\sqrt{t}} = \frac{1}{2} \qquad \dot{y} = 2t = 2 \qquad \dot{z} = -\frac{1}{t^2} = -1$$

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} = 1 \times \frac{1}{2} + 2 \times 2 - 2 \times 1 = 5/2$$

Example: If f(x,y,z)=2xy+z and x=cost, y=sint, z=t find $\frac{df}{dt}$ at t=0. H.W

Ex: let w=xyz =f(x,y,z)

x=rcost , y=r sint, z=r^2 find $\frac{df}{dr}$

Total Derivatives:

df(x,y)= $f_x dx + f_y dy$

Laplace equation:

A function f(x,y,z) is said to satisfy the 3D Laplace equation if

$$f_{xx} + f_{yy} + f_{zz} = 0$$

Ex: show that f(x,y,z)=$e^{3x+4y} cosz$ satisfy the 3D Laplace equation. H.W.

Ex: for what values of n the function f(x,y,z) =$(x^2 + y^2 + z^2)^n$ will satisfy 3D Laplace eq.

Directional Derivative:

Let w=f(x ,y). The direction derivative of f at a point$p_\circ$ $(x_\circ, y_\circ)$ in the direction of the point $p_1(x_1, y_1)$ is given by:

$$\frac{df}{ds} = \overrightarrow{grad} f(p_\circ).\vec{u}$$

Where grad f is called the gradient vector of f.

$$\overrightarrow{grad} f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

And $\vec{u}$ is a unit vector.

$$\vec{u} = \frac{p_{\circ}p_1}{|p_{\circ}p_1|}$$

Example: Find the directional derivative of f(x,y,z)=xy^2z^3

at the point$p_{\circ}$ (3,2,1)in the direction towards p_1(5,3,2)

$$\vec{u} = \frac{p_{\circ}p_1}{|p_{\circ}p_1|} = \frac{2i+j+k}{\sqrt{2^2+1^2+1^2}} = \frac{1}{\sqrt{6}}(2i + j + k)$$

$\frac{\partial f}{\partial x} = y^2z^3$ at $p_{\circ}$ (3,2,1) $\qquad \frac{\partial f}{\partial x} = 4$

$\frac{\partial f}{\partial y} = 2xyz^3$ at $p_\circ$ (3,2,1) $\qquad \frac{\partial f}{\partial y} = 12$

$\frac{\partial f}{\partial z} = 3xy^2z^3$at $p_\circ$ (3,2,1) $\qquad \frac{\partial f}{\partial z}$=36

$$\overrightarrow{grad} f(p_\circ) = 4i + 12j + 36k$$

$$\frac{df}{ds} = \overrightarrow{grad} f(p_\circ).\vec{u} = (4i + 12j + 36k).\frac{1}{\sqrt{6}}(2i + j + k) = \frac{56}{\sqrt{6}}$$

Exercise : 1-Find the directional derivative of f(x,y)=y$x^2 + xy^2$ at the point$p_\circ$ (1,1)in the direction towards p_1(2,4).

2- Find the directional derivative of w=f(x,y,z)= $z^3x + xyz$ at the point$p_\circ$ (1,0,1) in the direction of vector

A= 2i-j+2k.

Other properties and applications of gradient vector :

$$\nabla - \textbf{operater} = \overrightarrow{grad} f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Properties

f and g are two scalar functions

1. $\vec{\nabla}(f + g) = \vec{\nabla} f + \vec{\nabla} g$

2. $\vec{\nabla}(cf) = c\vec{\nabla} f$

3. $\vec{\nabla}(fg) = f\vec{\nabla} g + g\vec{\nabla} f$

Applications:

1- the normal line N.L $= \frac{x - x_\circ}{f_x} = \frac{y - y_\circ}{f_y} = \frac{z - z_\circ}{f_z}$

2- the tangent plane T.P. $f_x(x - x_\circ) + f_y(y - y_\circ) + f_z(z - z_\circ) = 0$

Divergence of a vector field:

If **F**(x ,y ,z)= $F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$ is avector field then the div .of this vector is

$$\text{Div}\vec{F} = \nabla.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Vector Functions

Def(vector functions): These are vectors whose components are functions

F(t) =X(t) i +Y(t) j +Z(t)k

This may represent a space curve or the motion of particle in space .

Ex: X(t) =cos t Y(t) =sin t Z(t) =-1

Note:

The line in a space is a special case of the vector function.

F(t) = ($x_\circ$+at)i +($y_\circ$+bt) j +($z_\circ$+ct)k

Derivative:

$$\frac{d}{dt}\,F(t) = \lim_{\Delta t \to 0} \frac{F(t-\Delta t) - F(t)}{\Delta t}$$

$$= X'(t)\, i + Y'(t)\, j + Z'(t)\, k$$

$$\frac{d}{dt}\,F(t) = \text{the tangent vector to the path}$$

Ex:

Find $\frac{d}{dt}F(t)\ if\ F(t) = tan^{-1}t\ i + \ln t\ j + e^{-t}\ k$

Sol:

$$\frac{d}{dt}F(t) = \frac{1}{1+t^2}\,i + \frac{1}{t}\,j - e^{-t}\mathrm{k}$$

At t=1 $\frac{d}{dt}F(t) = \frac{1}{2}i + j - e^{-1}k$

Velocity and acceleration:

Let **R(t)** the position vector given the position of moving particle in a space then

$\frac{d}{dt}\boldsymbol{R(t)} = \boldsymbol{V(t)}$ the velocity vector and $\frac{d}{dt}\boldsymbol{V(t)} = \boldsymbol{a(t)}$the acceleration

$$|\boldsymbol{V(t)}| = S(t) \text{ is speed scalar function of time}$$

$$|\mathrm{V}| = \sqrt{\mathrm{X}'(\mathrm{t})^2 + \mathrm{Y}'(\mathrm{t})^2 + \mathrm{Z}'(\mathrm{t})^2}$$

Find : a.. the velocity vector at $\mathrm{t} = \frac{\pi}{4}$

b..the acceleration at $\mathrm{t} = \frac{\pi}{4}$

a.. V(t) $=\frac{d}{dt}\boldsymbol{R(t)} = -3\sin t\, i + 3\cos t j + 2k$

$$\boldsymbol{V}\left(\frac{\pi}{4}\right) = \mathbf{-3\frac{1}{\sqrt{2}}}\boldsymbol{i} + \mathbf{3\frac{1}{\sqrt{2}}}\boldsymbol{j} + \mathbf{2}$$

b.. a(t) $=\frac{d}{dt}\boldsymbol{V(t)} = -3\, cost\, i$ $-3\sin \mathrm{t}\, \mathrm{j} + 0\mathrm{k}$

$$\mathrm{a}(\frac{\pi}{4}) = \mathbf{3\frac{1}{\sqrt{2}}}\boldsymbol{i} - \mathbf{3\frac{1}{\sqrt{2}}}\boldsymbol{j}$$

and S(t)= $\sqrt{\mathbf{9}\sin t^2 + \mathbf{9}\cos t^2 + 4} = \sqrt{\mathbf{13}}$

The length of the curve(distance traveled):

The length of the curve **R(t)** =X(t)+Y(t)+Z(t) from $t_\circ$ *to t is*

$$L \equiv \; s = \int_{t_\circ}^{t} \sqrt{X'(t)^2 + Y'(t)^2 + Z'(t)^2 \; dt}$$

$$= \int_{t_\circ}^{t} |\boldsymbol{V}(\boldsymbol{t})| \; dt$$

In this example take t=0 to 2π

$$s \equiv L = \int_0^{2\pi} \sqrt{13}\, dt = \sqrt{13}\,t \;\; = \left(\sqrt{13} \times 2\pi\right) - \left(\sqrt{13} \times 0\right) = 2\pi\sqrt{13}$$

The **TNB** system:

1. The unit tangent vector **T**: $\quad T = \dfrac{\frac{dR(t)}{dt}}{S(t)} = \dfrac{V(t)}{|V(t)|}$

2. The normal vector **N:** $\quad N = \dfrac{dT/dt}{|dT/dt|}$

3. The binormal vector **B** : $\quad B = T \times N$

Find T , N and B for the circular motion R(t) = cos2t i +sin2tj

Sol:

V(t)=-2sin2t i+2cos2tj and $|V(t)| = 2$

T= $\frac{V(t)}{|V(t)|} = \frac{-2\sin 2t\, i + 2\cos 2t j}{2}$ =- sin2t i+cos2tj

$dT/dt = -2\cos 2ti - 2sin2tj$

N=$\frac{dT/dt}{|dT/dt|} = -cos2ti - sin2tj$

B=T× $N = \begin{vmatrix} i & j & k \\ -sin2t & cos2t & 0 \\ -cos2t & -sin2t & 0 \end{vmatrix} = k$

T ,N ,B are orthogonal vectors i.e.

$$T.N=0 \;,\; N.B=0 \text{ and } T.B=0$$

$$|\mathbf{T}| = |\mathbf{N}| = |\mathbf{B}| = 1$$

Curvature: Rate change of the angle of the inclination Ø with respect to arc length s

.

$$K = \left|\frac{d\emptyset}{ds}\right|$$

And the radius of curvature is $\rho = \frac{1}{k}$

Theorem $K=\left|\frac{d\boldsymbol{T}}{ds}\right|$

:

a. if the curve is given y=f(x)

$$K=\frac{Y''}{[1+Y'^2]^{3/2}}$$

b. if the curve is given x=g(y)

$$K=\frac{\frac{d^2x}{dy^2}}{1+(\frac{dx}{dy})^2}$$

c. if the curve is given y=f(t) and x=g(t)

$$K=\frac{|xy-yx|}{[x^2+y^2]^{3/2}}$$

or

$$K=\frac{|V\times a|}{|V|^3}$$

from which we can find K for a space curve find the curvature of the curves

R(t)=a cost i+a sin t j   at any time t

sol:

R= a cost i+a sin t j

V= - a sin t i+ a cos t j

a=-a cos t i- a sin t j

$$K=\frac{|V\times a|}{|V|^3} \qquad V\times a = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -\text{a}\sin t & 0 \end{vmatrix} = a^2 k$$

$$|V| = \sqrt{a^2\sin t^2 + a^2\cos t^2} = a$$

$$=\frac{|V\times a|}{|V|^3} = \frac{a^2}{a^3} = \frac{1}{a}$$

The Complex numbers

Def: The order pair z=(x, y) where x and y are real numbers is called the complex number.

Notations:

1. The complex number (0,y) is called pure imaginary number.
2. The real number x is called the real part of z and The real number y is called the imaginary part of z.
3. We say that the complex numbers(x_1, y_1) and(x_2, y_2) are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.
4. The add ion and the multiplication are defined as: $z_1 + z_{2=} (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

   $z_1 . z_{2=}(x_1, y_1).(x_2, y_2) = (x_1x_2 - y_1y_2 , x_1y_2 + x_2y_1)$

1. If i= $\sqrt{-1}$ then we can write the complex number z = x+iy and
   $z_1 . z_{2=}(x_1, y_1).(x_2, y_2) = (x_1x_2 - y_1y_2 , x_1y_2 + x_2y_1)$

   $z_1 + z_{2=}(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Examples:

1. (2+3i)+(1+4i)=(2+1)+ (3+4)i=3+7i
2. (1−i).(2+3i)=(1×2 –(-1× 3) $+ (1 \times 3 + (2 \times (-1))$i=5+i
3. $i^3 = (i^2)$ i=−i

Algebraic properties :

1.The commutative law : $z_1 + z_2 = z_2 + z_1$ and $z_1 . z_2 = z_2 . z_1$ $\forall\ z_1, z_2 \epsilon ₵$

2. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$z_1 . (z_2 . z_3) = (z_1 . z_2) . z_3$

4.The additive identity is 0=0+0i then $\forall z\epsilon ₵$,z+0=0+z=z

5.The multiple identity is 1=1+0i then $\forall z\epsilon ₵$,z.1=1.z=z

6.The additive inverse $\forall z\epsilon ₵ \ \exists -z=-x-\mathrm{i}y$ st .z+(-z)= −z+ z=0

7. The multiplicative inverse $\forall z\epsilon ₵ \ \exists z^{-1}$ s t. z.$z^{-1}=z^{-1}.z=1$

8. the conjugate of the complex number z=x+ i y is z^{-} = x− i y

Examples:

1.(6+5i)-(4-3i)+(2+7i)=4+15i

3.($\sqrt{2}$-i)−i(1-$\sqrt{2}$ i)=($\sqrt{2}$-i)−(i $+\sqrt{2}$)=-2i

4The conjugate of 3-7i= 3+7i

6. find the inverse of −2+3i

$$z^{-1}=\frac{1}{z}=\frac{1}{-2+3i}=\frac{1}{-2+3i}\times\frac{-2-3i}{-2-3i}=\frac{-2-3i}{13}=\frac{-2}{13}+\frac{-3i}{13}$$

Graphical representation of the complex number:

Every complex number z=x+ iy corresponding one point in the plane XY For example (0,0) corresponds to the complex number z=0+0i and the number z represents the distance from (0,0) to (x ,y) therefore the plane is called the complex plane ,X is called the real axis and Y is called the imaginary axis. Y

The absolute value of the complex number:

The absolute value of the complex number z=x+ i y is defined as fellows:

$$|z| = \sqrt{x^2 + y^2}$$

Note:

1. The number z represents the distains between the origin and (x, y)
2. If $z_1 = (x_1, y_1)$and $z_2 = (x_2, y_2)$ then the distance between them is

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$3. |z_1 . z_2| = |z_1| . |z_2|$

$4. \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad z_2 \neq 0$

Example:

z= 2-3i then

$|z| = \sqrt{x^2 + y^2} = \sqrt{2^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$

polar form the complex number:

Let r, θ are the polar coordinates correponding to (x, y)that represents z

x=r cos θ , $y = r \sin \theta$

$z = r(\cos \theta + i\sin\theta) = re^{i\theta}$

s .t . $r = |z| = \sqrt{x^2 + y^2}$ and θ it is the angle of the complex number z

, it is called (argument)andcan be write $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

Examples:

1. write z=1+i by the polar form :

sol:

r= $\sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2 = \sqrt{2}}$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$z = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right) = e^{i\frac{\pi}{4}}$$

1. $z = i$

$$r = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

z= $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ $= e^{i\frac{\pi}{2}}$

seince $\cos\theta$ and $\sin\theta$ are pirodic in 2π then $\arg z = \theta + 2k\pi$

if k=0,argz= θ $\quad , -\pi < \theta < \pi$

notations:

1. $\arg(z_1 + z_2) = \arg z_1 + \arg z_1$
2. let $z_1 = r(\cos\theta + i\sin\theta)$ $\quad , z_2 = p(\cos\emptyset + i\sin\emptyset)$

$z_1 . z_2$
$= rp(\cos\theta + i\sin\theta).(\cos\emptyset + i\sin\emptyset) \quad = rp(\cos\theta\cos\emptyset + i\sin\theta\sin\emptyset) = rp(\cos(\theta$

$\arg(z_1 . z_2) = \theta + \emptyset$

1. $\arg \frac{z_1}{z_2} = arg\ z_1 - arg z_2$
2. for all the integer number n $z^n = r^n(\cos n\ \theta + isin\ n\theta) = r^n(\cos\theta + i\sin\theta)^n$

Example: represent the following complex numbers in the standard form:

z=$e^{i\theta}$ $\rightarrow$ $r = 1\ \ and\ \theta = \frac{\pi}{2}$

X=r cos$\theta = 1 * cos\frac{\pi}{2} = 0$ $\quad$ $y = r\ sin\theta$=1.sin$\frac{\pi}{2} = 1 * 1 = 1$

Z=x+iy=0+i.

The complex function:

Let S be non empty set of the points in the complex plane if ∀ $z \epsilon S$ ∃ w

s.t. w=f(z). i.e. f: S→ ₵ , S is called domain f and f(z) is called the range.

We can write f(z) by the following:

w=f(z) =u(x ,y)+iv(x ,y) ,u ,v are real functions.

Example:

1. f(z)=$x^2 + 2y - \text{i}2xy^3$,, u(x, y)= $x^2 + 2y$,v$(x, y) = -\text{i}2xy^3$
2. f(z)= z^2 write f(z) by u and v.

sol: z=x +i y → f(z)=$(x + iy^2)$

f(z)= $x^2 + 2\text{i}xy - y^2$ =$x^2 + y^2 - 2\text{i}xy$

u(x, y)= $x^2 + y^2$, v$(x, y) = -2\text{i}xy$